How Invariant is the Measured Equation of Invariance?

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***Abstract*—We prove that the measured equation of invariance (MEI) is not invariant to the excitation. The proof is based on the fact that different sets of sinusoidal metrons produce different boundary equations, even when the nodal separation is infinitely small. These results follow from the rigorous analytical derivations which are also verified numerically. The noninvariance of the MEI emphasizes the importance of the proper choice of metrons and indicates that this choice should be influenced by the excitation and geometry in question.**

I. Introduction

The MEASURED equation of invariance (MEI) has been introduced by Mei *et al.* [1] as a novel boundary truncation technique for use in finite element and finite difference methods. The MEI is constructed using a set of outgoing fields that originate from the currents flowing on the surface of the scatterer. These currents are called metrons. An analysis of the MEI, with detailed derivations, discussions, and numerical examples can be found in [2] and [3].

It is postulated in [1] that the MEI is invariant to the excitation or, equivalently, to the particular set of metrons used. If this is true, it would not matter what metrons are used as long as they permit us to find the MEI. Thus, we would have a very robust way to construct a boundary condition, independent of the excitation in question. However, the numerical results in [4] and [5] suggest that this is not true. Indeed, we will prove that the above mentioned postulate is incorrect.

The MEI is analyzed on a circular boundary. We simplify the results by considering an electrically large cylinder and using the asymptotic form of the MEI. This is derived analytically and tested numerically. Two different sets of sinusoidal metrons are used, and they result in two different boundary equations. In fact, we show that they are different even in the limit as the nodal separation goes to zero.

II. The MEI Formulation

The geometry is shown in Fig. 1. A TM polarized wave, whose wavenumber equals k, is normally incident on a perfectly conducting circular cylinder of radius a. A regular finite difference (FD) grid is constructed such that the radial and angular spacings between neighboring nodes are respectively $\Delta\rho$ and $\Delta\phi$. The radius of the outer mesh boundary is b. The MEI equation for node number 6 is given by:

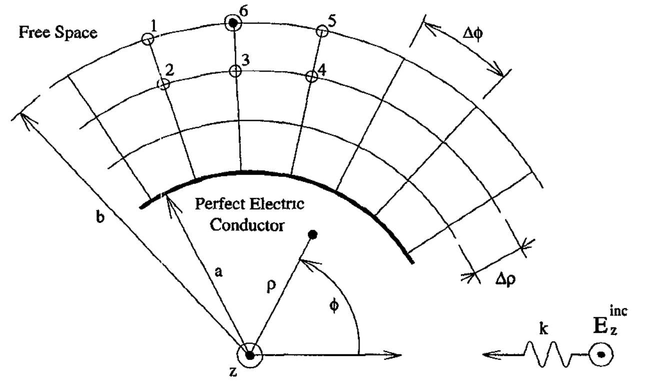

Fig. 1. Perfectly conducting circular cylinder geometry for analysis of the MEI.

$$\sum_{i=1}^{6} a_i E_z^s(\rho_i, \phi_i) = 0 \qquad (1)$$

Here E_z^s is the scattered electric field, and ρ_i and ϕ_i are the coordinates of node i.

According to the MEI method we need at least 5 metrons to determine the MEI coefficients a_i. The coefficients must be chosen such that the MEI is exactly satisfied for all fields radiated by the metrons. Thus, if we choose the lowest order sinusoidal currents for metrons:

$$e^{-j2\phi}, e^{-j\phi}, 1, e^{j\phi}, e^{j2\phi} \qquad (2)$$

the MEI (1) must satisfy:

$$\sum_{i=1}^{6} a_i H_n^{(2)}(k\rho_i) e^{jn\phi_i} = 0 \quad n = -2, -1, 0, 1, 2 \qquad (3)$$

One coefficient is always arbitrary, so let us set the sixth one to 1. Furthermore, the form of the above system of equations indicates that the coefficients are symmetric. Therefore, we can write:

$$\begin{aligned} a_4 &= a_2 \\ a_5 &= a_1 \\ a_6 &= 1 \end{aligned} \qquad (4)$$

We have only three unknown coefficients left: a_1, a_2, and a_3.

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III. Asymptotic Analysis For Electrically Large Cylinder

We are now in position to solve for the MEI coefficients. However, the analytic result is not very enlightening due to its great complexity. Much more insight is gained if we use the asymptotic form of the terms appearing in (3) when the electrical radius kb of the outer mesh boundary is large. These expansions can be found in [6]. Even when the asymptotic approximations are used, the solution of (3) is best left to one of the computer programs for symbolic manipulation. MAPLE [7] is used to obtain the following solution:

$$\left.\begin{aligned} a_1 &= -\frac{1}{2}\frac{3+(\alpha_r k\Delta\rho)^2 + j3k\Delta\rho}{3-5(\alpha_r k\Delta\rho)^2 + j3k\Delta\rho} \\ a_2 &= \frac{1}{2}e^{-jk\Delta\rho}\frac{3+(\alpha_r k\Delta\rho)^2 - j3k\Delta\rho}{3-5(\alpha_r k\Delta\rho)^2 + j3k\Delta\rho} \\ a_3 &= -e^{-jk\Delta\rho}\frac{3-5(\alpha_r k\Delta\rho)^2 - j3k\Delta\rho}{3-5(\alpha_r k\Delta\rho)^2 + j3k\Delta\rho} \end{aligned}\right\} + O(\frac{1}{kb}) \quad (5)$$

where α_r stands for the aspect–ratio of the FD cell defined by:

$$\alpha_r = \frac{b\Delta\phi}{\Delta\rho} \quad (6)$$

A detailed derivation of this result can be found in [3].

The asymptotic form (5) can be compared to the actual coefficients obtained in the implementation of the MEI method. The magnitude of the error is shown in Fig. 2 versus the electrical size of the outer mesh boundary. The mesh has a nodal density of 20 nodes per wavelength. The inclination of the curves in this log–log plot ranges from 0.9891 to 0.9916 indicating that the error is approximately proportional to $1/kb$, which agrees with the result in (5).

IV. Equivalent Differential Equation

We now investigate the limiting case when the size of the FD cell goes to zero. To this end, let us assume that some field $F(\rho,\phi)$ satisfies the MEI boundary condition (1). With reference to Fig. 1 this can be written as an ordinary finite difference equation in polar coordinates:

$$\begin{aligned} \mathbf{B}F =& F(b,0) + a_1[F(b,-\Delta\phi) + F(b,\Delta\phi)] \\ &+ a_2[F(b-\Delta\rho,-\Delta\phi) + F(b-\Delta\rho,\Delta\phi)] \\ &+ a_3F(b-\Delta\rho,0) = 0 \end{aligned} \quad (7)$$

To determine the differential equation associated with this difference equation, we should replace the values of the field

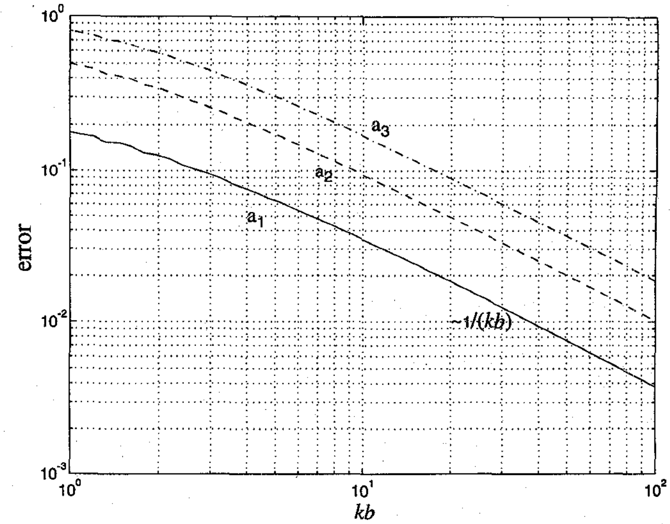

Fig. 2. Plot of the magnitude of the error in the asymptotic expansion for the MEI coefficients as a function of the electrical radius kb of the outer mesh boundary (20 nodes per wavelength, the lowest order sinusoidal metrons).

F at different nodes with its Taylor expansion centered at node 6. We should also have in mind that the MEI coefficients a_i depend on the cell size according to (5), which allows us to write (7) in the following form:

$$\mathbf{B}F = -2\alpha_r^2(k\Delta\rho)^3\mathbf{L}F + O(\Delta\rho^4) = 0 \quad (8)$$

where $\mathbf{L}$ is a linear differential operator given by:

$$\mathbf{L} = \frac{\partial}{\partial(k\rho)} + j + j\frac{3}{4(kb)^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{4(kb)^2}\frac{\partial^3}{\partial(k\rho)\partial\phi^2} \quad (9)$$

Therefore, $\mathbf{L}$ is the differential equation that is equivalent to the MEI when the cell size is infinitely small.

A detailed derivation of this result and some of its implications are given in [3].

V. A Different Set of Metrons

All the results derived so far correspond to the choice of metrons given by (2). What if a different set of metrons is used? For example:

$$e^{-jkb\phi}, e^{-j\phi}, 1, e^{j\phi}, e^{jkb\phi} \quad (10)$$

The first and the last metron in this set correspond to the sinusoidal current that has the same wavelength in the ϕ direction as the incident wave. These metrons are chosen for no particular reason except to simplify the results. Namely,

$$\left.\begin{aligned} a_1 &= -\frac{1}{2}\frac{1-cos(\alpha_r k\Delta\rho) + j\alpha_r^2 k\Delta\rho(e^{jk\Delta\rho}-1)}{1-cos(\alpha_r k\Delta\rho) + j\alpha_r^2 k\Delta\rho(e^{jk\Delta\rho}-1)cos(\alpha_r k\Delta\rho)} \\ a_2 &= \frac{1}{2}\frac{1-cos(\alpha_r k\Delta\rho) + j\alpha_r^2 k\Delta\rho(1-e^{-jk\Delta\rho})}{1-cos(\alpha_r k\Delta\rho) + j\alpha_r^2 k\Delta\rho(e^{jk\Delta\rho}-1)cos(\alpha_r k\Delta\rho)} \\ a_3 &= -\frac{1-cos(\alpha_r k\Delta\rho) + j\alpha_r^2 k\Delta\rho(1-e^{-jk\Delta\rho})cos(\alpha_r k\Delta\rho)}{1-cos(\alpha_r k\Delta\rho) + j\alpha_r^2 k\Delta\rho(e^{jk\Delta\rho}-1)cos(\alpha_r k\Delta\rho)} \end{aligned}\right\} + O(\frac{1}{(kb)^{1/3}}) \quad (11)$$

we can follow exactly the same steps that were outlined in the previous sections.

For the asymptotic form of the MEI coefficients we obtain (11), shown at bottom of previous page, and for the equivalent differential equation we derive:

$$\mathbf{L} = \frac{1}{2}[\frac{\partial}{\partial(k\rho)} + j + j\frac{1}{(kb)^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2(kb)^2}\frac{\partial^3}{\partial(k\rho)\partial\phi^2}]. \quad (12)$$

VI. Conclusion

The measured equations of invariance resulting from two different sets of metrons, (2) and (10), are compared. We found that the MEI coefficients have different asymptotic forms in each case, (5) and (11). Even when the nodal separation goes to zero, these two choices of metrons result in two different boundary conditions, as demonstrated by (9) and (12). Thus, we can conclude that the MEI is not invariant to the choice of metrons, and consequently, the postulate of invariance must be incorrect. Furthermore, (8) indicates that the residual of the MEI $\mathbf{B}F$ equals zero to within third order accuracy with respect to the FD cell size. However, the differential equation $\mathbf{L}F = 0$ is enforced only to the first order accuracy. The results indicate that an optimum choice of metrons depends on the particular excitation and geometry in question.

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